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Admittance of Infinite and Finite Cylindrical Metallic Antenna

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A monopole antenna excited by a coaxial line is studied in detail. The mathematical model of the excitation is a magnetic ring current the radius of which may be larger than the radius of the cylinder. First a study is made of the waves guided by and radiated from an infinite cylinder. Numerical results for the current and the conductance are given. The conductance decreases when the gap width increases.

The standing-wave antenna is treated with an assumed effective gap at the antenna ends. A solution of the antenna problem is found by superimposing the multiply reflected waves. In the solution, the effect of the gap and the antenna end separates. Accurate values for the gap admittance and the end admittance are derived from experimental values obtained by Hartig.

1. Introduction

The subject of metallic cylindrical antennas has been treated extensively in the past. It suffices here to refer to recent treatments and reviews by King (1956) and Duncan and Hinchey (1960). As soon as the assumption of an infinitely thin antenna is abandoned, a great deal of complexity arises because a finite cylinder cannot be treated in a simple way as a boundary value problem in electrodynamics. Even if this were possible there is still the problem of coupling the isolated cylinder in a meaningful way to a transmission line. This problem is generally known as the gap problem. Almost all existing theories use a delta-function gap which results in an infinity in the susceptance and this infinity is then circumvented in a more or less arbitrary manner (see e.g., Duncan, 1962; Hallén 1953, 1962).

Theories like King-Middleton's second-order theory that forces the current to be a modification of a sinusoidal distribution right into the gap imply a gap of nonzero width. It is often stated that values extrapolated from experiments with decreasing values of gap width should be compared with zero-width theory. Actually the limit does not exist and we see later that King-Middleton's second-order theory implies a width of the order of the radius of the antenna.

Solutions based on iterative solutions to an integral equation, Fourier-series solutions, and direct numerical solutions using large computers have their advantages but are difficult to interpret physically. When this is the case, the separation of the source problem from the antenna-end problem is almost impossible; to foresee what happens is difficult when the antenna is placed in another environment such as a dielectric coating.

The concept adhered to in this paper is that of traveling waves along the cylinder. These waves are most easily studied on the infinite cylinder that represents a solvable boundary value problem. To take into account the transmission line antenna coupling, the antenna model chosen is a monopole over a ground plane where the cylindrical antenna is a continuation of the inner conductor in the feeding coaxial line. The true source for the radiated field is then an incoming TEM wave in the coaxial line. From the opening in the ground plane the energy radiates under the influence of the cylinder. First the infinite cylinder with no reflections is treated and its admittance is discussed. The waves along the cylinder are slow and are slightly damped by radiation losses. The ends are treated as effective gaps where the effective gap voltage is determined from a semi-infinite problem solution (Hallén, 1956) for a thin-walled tube and derived indirectly from experimental data. The total solution for the admittance is then expressed in terms of a multiple scattering between the coaxial line and the end. In this way a simple solution is obtained that separates the effect of the end and the gap. A similar solution for the admittance has been obtained in another way by Hallén (1953, 1962) for the delta-function gap. Hallén's solution suffers though from the usual ambiguity in susceptance because an infinitely narrow gap is used.

2. Infinite Antenna

2.1. Far Field From Magnetic Ring Source

The physical model adopted for the infinite antenna (fig. 1a) consists of an infinite, straight, and perfectly conducting cylinder excited by a coaxial transmission line in which the inner conductor has the same radius as the antenna. The ground plane is assumed to be of infinite extent. This excitation corresponds quite closely to a practical case and has been used by Hartig (referred to by King, 1956) and Altshuler (1961), so the theoretical results can be readily compared with experimentally obtained values.

The opening in the ground plane at the end of the coaxial line can be considered as the antenna gap of width $w=b-a$ and mean radius $r_m=\frac{1}{2}(b+a)$. In accordance with the usual theory of slot antennas, the fields in the aperture (or gap) can be replaced by a distribution of electric and magnetic currents in the aperture, if the gap width is small ($w \ll \lambda$), the magnetic currents will dominate. Thus we get the mathematical model shown in figure 1b which we shall use for finding the far field from the gap. The magnetic ring current density is assumed to be of the form

$$\overline{K}=-2V\delta(r-r_s)\delta(z)\hat{\phi}, \tag{1}$$

where V is the voltage in the gap.

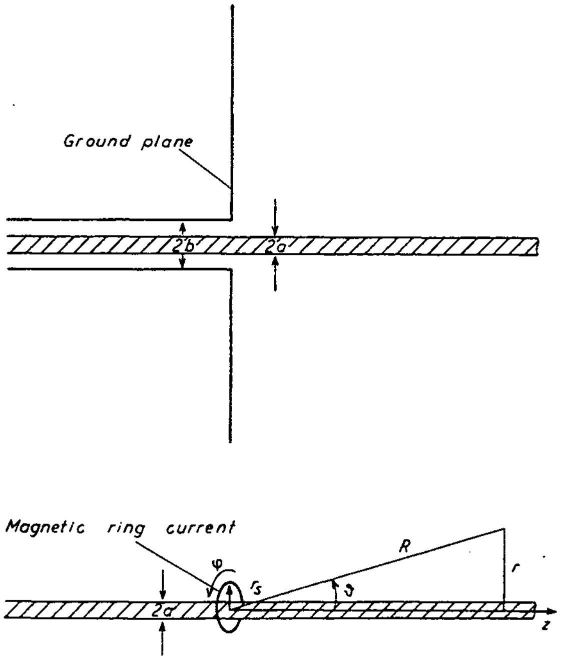

FIGURE 1. *Models of infinite antenna.*

(a) Physical model is a semiinfinite monopole excited by a coaxial line. (b) Mathematical model is an infinite rod excited by a magnetic ring current at radius r_s.

This model of the gap is used only in this section where the far field is found, it is inadequate for determining the current near the gap.

The problem is solved by a Fourier-transform method

$$\zeta_0 h_\phi(r,\phi,h)=\int_{-\infty}^{+\infty}\zeta_0 H_\phi(r,\phi,z)e^{-ihz}dz, \tag{2}$$

and similarly for the other field-components. A time factor exp $(-i\omega t)$ is used throughout.

In the following we concentrate on the H_ϕ component both at the surface of the rod and far from the rod. For the transformed quantity $\zeta_0 h_\phi$ the following expressions are found by a standard technique

$$r=a: \qquad \zeta_0 h_\phi=-i\frac{2Vk_0 r_s}{\chi a}\frac{H_0^{(1)\prime}(\chi r_s)}{H_0^{(1)}(\chi a)}, \tag{3}$$

$$r>r_s: \qquad \zeta_0 h_\phi=-V\pi k_0 r_s\frac{H_0^{(1)\prime}(\chi r)}{H_0^{(1)}(\chi a)}\left[J_0(\chi a)H_0^{(1)\prime}(\chi r_s)-J_0'(\chi r_s)H_0^{(1)}(\chi a)\right], \tag{4}$$

where $\zeta_0=\sqrt{\frac{\mu_0}{\epsilon_0}}=120\pi$ ohms,

$k_0=\frac{2\pi}{\lambda}$, free-space wave number,

χ is defined by $\chi^2=k_0^2-h^2$.

To find $\zeta_0 H_\phi$, we need the inverse transform

$$\zeta_0 H_\phi=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\zeta_0 h_\phi e^{ihz}dh. \tag{5}$$

An asymptotic expression for the magnetic field along the rod $(r=a)$ is found via a branch-cut integration with the following result

$$\zeta_0 H_\phi\simeq\frac{2V}{a}e^{ik_0 z}f_2(z/\ell), \qquad z\gg\ell, \tag{6}$$

where

$$f_2(x)=\int_0^\infty\frac{e^{-t}dt}{t\left[\left(\ln\frac{\Gamma t}{ix}\right)^2+\pi^2\right]} \tag{7}$$

and $\ell=\frac{k_0 a^2\Gamma}{2}$.

It is worth noting that the r_s dependence has disappeared in this approximation; this means that for the field sufficiently far out along the rod, the radial position of the source is of no importance. It is therefore not surprising that (6) is identical with expressions for the current on an infinite antenna of the usual idealized model of a thin-walled tube with an infinitesimally narrow circumferential gap (Hallén, 1948; Vainshtein, 1959a, b; Chen and Keller, 1962; Duncan, 1962; Kunz, 1963).

The total current through the rod is

$$I = 2\pi a H_\phi = \frac{4\pi V}{\zeta_0} e^{ik_0 z} f_2(z/\ell) = Y_w V e^{ik_0 z}, \tag{8}$$

where a wave admittance Y_w has been introduced

$$Y_w(z) = \frac{4\pi}{\zeta_0} f_2(z/\ell). \tag{9}$$

The integral has been evaluated asymptotically by several authors. The most accurate series has been given by Kunz (1963) as

$$\frac{\zeta_0}{4\pi} Y_w(z) = \frac{1}{\ln \frac{iz}{\ell\Gamma}} - \frac{\ln \Gamma}{\left(\ln \frac{iz}{\ell\Gamma}\right)^2} - \frac{1.3118}{\left(\ln \frac{iz}{\ell\Gamma}\right)^3} + \frac{0.2520}{\left(\ln \frac{iz}{\ell\Gamma}\right)^4} + \frac{3.9969}{\left(\ln \frac{iz}{\ell\Gamma}\right)^5}. \tag{10}$$

If the first two terms in this series are combined to one, we find the correct first-order term for a very thin antenna or for z very large

$$Y_w(z) \sim \frac{4\pi}{\zeta_0} \frac{1}{\ln \frac{iz}{\ell}}. \tag{11}$$

Equation (11) is apparently valid also for small z if ℓ is sufficiently small, but it should be used with care. If kz is too small (< 0.5) the approximations made are too inaccurate. A plot of the complex function Y_w is given in figure 2; the results are found by numerical integration of (7) and are compared with Kunz' series, eq (11).

The current wave on the infinite rod decreases slowly with distance and it is damped by radiation losses. The phase velocity is slightly smaller than the free-space velocity

$$\frac{c}{v_f} \sim 1 + \frac{\pi}{2\left(\ln \frac{z}{\ell}\right)^2 \cdot k_0 z} \tag{12}$$

for $z \gg \ell$.

The slow current wave of which (11) is only the leading term is essentially bound to the cylinder over a long distance but is eventually radiated.

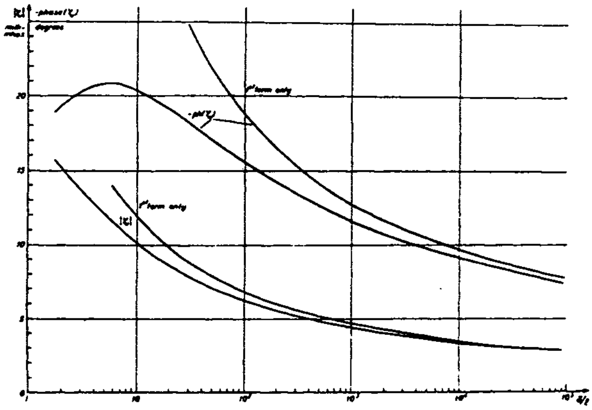

FIGURE 2. *Complex wave admittance as a function of normalized distance along the infinite rod,* $\ell = \frac{k_0 a^2 \Gamma}{2}$.

The radiated far field is found by applying the inverse Fourier transform to (4) and using a saddle-point technique, we get

$$\zeta_0 H_\phi = -V f_1(\theta) \frac{e^{ik_0 R}}{R}, \tag{13}$$

where
$$f_1(\theta) = k_0 r_s i \frac{J_0(x) Y_0'(y) - J_0'(y) Y_0(x)}{H_0^{(1)}(x)}, \tag{14}$$

and $x = k_0 a \sin\theta$, $y = k_0 r_s \sin\theta$.

For small θ or small $k_0 a$, $k_0 r_s$, we can expand the Bessel and Hankel functions

$$\zeta_0 H_\phi \sim -\frac{V}{\sin\theta \ln \dfrac{k_0 a \sin\theta\Gamma}{2i}} \frac{e^{ik_0 R}}{R} \tag{15}$$

for x and $y \ll 1$.

Again, the radial position of the magnetic ring current does not enter in the equation in this approximation.

2.2. Admittance of Gap-Excited Infinite Antenna

The preceding sections indicate that insofar as the far field is concerned, the idealized mathematical model of an impressed voltage over a small gap is sufficient. This is, however, not the way antennas are usually excited and the true source for the radiated energy is the TEM wave in the feedline, in this case the coaxial line. The radiation influence on the distribution of current and voltage on the transmission line is established via the impedance concept. When the impedance has been found at a proper plane of reference (chosen here at a distance from the gap equal to an integral number of half wavelengths), the antenna can be treated as a circuit element. The admittance of the infinite antenna excited by a coaxial line has been considered by Papadopoulos (1960), who gives some numerical results for a restricted range of parameters; his results disagree with those obtained here. Wu (1962, 1963) also presents a solution but no quantitative results since an integral equation remains to be solved. The recent results obtained by Otto (1967) are in good agreement with those in section 3.2.

2.2.1. Conductance of Infinite Antenna

The conductance of the infinite antenna, G_∞, is defined by

$$\tfrac{1}{2} G_\infty V^2 = W_r \tag{16}$$

where W_r is the total power radiated and V is the voltage in the transmission line at a proper plane of reference mentioned above. We now make the assumption that V equals the effective V in the gap, V_{gap}, which was used in the preceding sections. It amounts to neglecting higher order modes in the transmission line and as the total power radiated is rather insensitive to the size of the gap the same is probably true for the conductance. The radiated power can be expressed in terms of the magnetic far field

$$W_r = \frac{1}{2}\zeta_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} |H_\phi|^2 R^2 \sin\theta \, d\theta \, d\phi, \tag{17}$$

or from (13)

$$G_\infty = \frac{2\pi}{\zeta_0} (k_0 r_s)^2 \int_{\theta=0}^{\frac{\pi}{2}} \left| \frac{J_0(x)Y_0'(y) - J_0'(y)Y_0(x)}{H_0^{(1)}(x)} \right|^2 \sin\theta d\theta. \tag{18}$$

For a thin antenna, a series expansion of G_∞ has been given by Hallén (1948) and a numerical investigation in the general case by Duncan (1962). For $x, y \ll 1$, (18) reduces to

$$G_\infty = \frac{2\pi}{\zeta_0} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta \left| \ln \frac{ka \sin\theta\Gamma}{2i} \right|^2}, \tag{19}$$

which for ka small can be integrated to give

$$G_\infty = \frac{2\pi}{\zeta_0} \left(1 - \frac{2}{\pi} \arctan \left(\frac{2}{\pi} \ln \frac{2}{ka\Gamma} \right) \right), \tag{20}$$

or if only the leading term is retained

$$G_\infty \sim \frac{2\pi}{\zeta_0} \frac{1}{\ln \frac{2}{ka\Gamma}}. \tag{21}$$

Numerical values based on (18) are shown in figure 3. As was anticipated G_∞ is rather insensitive to variations in r_s as long as $k_0 r_s$ is small. For a thick antenna, G is influenced substantially by $\tau = r_s/a$. This means that the conductance of a gap will decrease with increasing gap width and more so the thicker the antenna is.

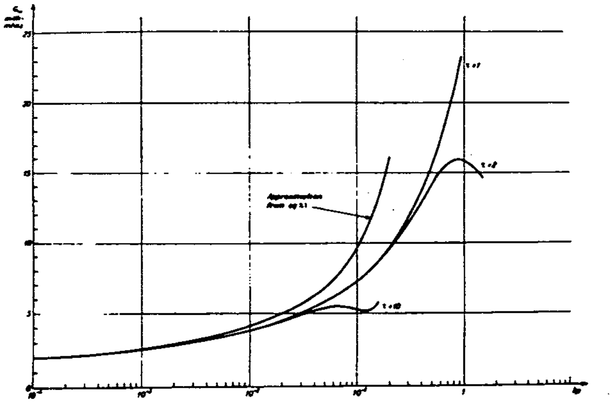

FIGURE 3. *Conductance of infinite antenna excited by magnetic ring current at* $r = r_s = \tau a$.

2.2.2. Susceptance of Infinite Antenna

The susceptance corresponds to the stored energy in the near field, in this case the stored energy in the capacitor formed by the ground plane and the infinite rod. The complete discontinuity problem must be solved to find the susceptance. This is not done here but an approximation to the susceptance can be derived from the static calculation of a similar problem.

In Whinnery and Jamieson (1944, eq. 8) the exact discontinuity capacitance per unit length for a step in a parallel plate transmission line is derived

$$C_d(\alpha)=\frac{\epsilon_0}{\pi}\left(\frac{\alpha^2+1}{\alpha}\cosh^{-1}\frac{1+\alpha^2}{1-\alpha^2}-2\ln\frac{4\alpha}{1-\alpha^2}\right), \tag{22}$$

where α is the ratio of the smaller to the larger distance between the plates. For a small gap compared with radius, the capacitance of the coaxial-line discontinuity is found by multiplying $C_d(\alpha)$ by the circumference $2\pi a$. The value of α is apparently zero but we can argue that the stored energy must be confined approximately within a wavelength from the gap, so a value of α equal to $(b-a)/\lambda$ seems reasonable. Thus we end with the following for the susceptance

$$B_\infty=\omega C_d=-\frac{4k_0a}{\zeta_0}\ln\frac{\lambda e^2}{4(b-a)}, \tag{23}$$

where (22) has been expanded for small α. The minus sign stems from our choice of time factor. A similar but probably more accurate expression is given in "Waveguide Handbook" (1951, sec. 4.7a, Parallel-plate guide radiating in half space, E plane) from which we get for the coaxial line with infinite center conductor

$$B_\infty=-\frac{4k_0a}{\zeta_0}\ln\frac{\lambda e}{4\Gamma(b-a)}, \tag{24}$$

$\Gamma=1.781$, $e=2.718$.

In section 3.2 precise values for the gap susceptance are derived from experimental data. It turns out that even (24) is not a very accurate expression for practical values of b/a. It is wrong by a factor varying between 1.5 and 2, but the form of the variation as a function of ka or $b-a$ is correct. The effect of a large gap on the field distribution inside the transmission line has been calculated by King (1955); a correction for narrow gap has not been made.

For k_0a fixed B_∞ goes to infinity as $b-a\rightarrow 0$, so the magnitude of the current in the gap is very large relative to the current on the rod for the slice-generator which is the usual model for a source-excited antenna. For a fixed separation B_∞ tends to zero as $k_0a\rightarrow 0$. A similar problem for a center-driven cylindrical antenna has been discussed by Wu and King (1959).

There still remains an ambiguity in the theory of center-driven antennas with a delta-function generator. An infinite correction should be made to get a finite current at the gap and it is a matter of choice how this is achieved. Duncan (1962) chooses to extrapolate the slowly varying part of the current right into the gap and Hallén (1953) uses the current at a distance from the gap approximately equal to the radius.

The only unambiguous way to treat the problem is by considering the antenna and transmission line as one unit as is done in the model used here. A measurable quantity is the relative susceptance, B_∞/Y_0, where Y_0 is the characteristic admittance of the transmission line. The value B_∞/Y_0 is not singular when the gap width tends to zero; in fact, it is zero.

3. Finite, Standing-Wave Antenna

3.1. The End Admittance

At points not too close to the end, the reflected wave obviously cannot be distinguished from a wave originating from a source on the infinite cylinder, and the position of the source must be at the end of the cylinder for physical reasons. There is of course no source of energy at the end but the introduction of an effective source is convenient both for the mathematical formulation and for the physical understanding; the difficulty lies in finding the strength of the effective source which must depend on the form of the end. The reasoning resembles to some extent the reasoning behind the geometrical theory of diffraction with waves (not rays) emanating from the edge; in our case, however, the wavelength is comparable to or much larger than one of the linear dimensions, so the theory is not applicable. In the theory of cylindrical, linear antennas, the end diffraction can be considered as a canonical problem corresponding to the half-plane problem in the geometrical theory of diffraction. Once it is solved the end diffraction can be treated approximately as a local phenomenon giving rise to a reflected wave of known amplitude.

Fortunately this important problem has been solved by Hallén (1956) for a particular type of end, the end of a thin-walled tube. We see later how the end problem can be solved experimentally.

Introducing the concept of gap voltage, V_e, for the effective source at the end, we can define the end-admittance Y_e as

$$Y_e = \frac{I_r(z=L)}{V_e} \tag{25}$$

where I_r is the reflected current wave. As a rough approximation the end admittance should equal the gap admittance for the infinite antenna Y_∞, at least for the conductance part, because G_∞ is fairly independent of the form of the gap. The susceptance B_∞ was determined by the static capacitance around the gap and similarly B_e must depend on the actual form of the end. The best one can do theoretically at present is to use Hallén's (1956) results for the tube ending which are probably typical for a large class of ends. The value Y_e has been plotted by Hallén for a large range of values of ka, so Y_e can be considered as known. For a very thin tube, Hallén finds

$$Y_e \sim \frac{2\pi}{\zeta_0} \frac{-1}{\ln (ka\Gamma) - i\frac{\pi}{2}}, \tag{26}$$

the real part is in good agreement with G_∞ (21). For other values of z than $z=L$ the reflected wave is given by the usual asymptotic expression, (8). By imposing the boundary condition $I=0$ at $z=L$, we can find

$$V_e = \frac{-I_i(z=L)}{Y_e}, \tag{27}$$

where I_i is the incident current wave.

In the following sections we discuss the effect of the reflected wave on the admittance of the antenna.

3.2. Admittance of Finite Antenna

As mentioned previously the admittance is only uniquely defined inside the transmission line, we must, therefore, evaluate the effect of the reflected wave in the transmission line. The reflected wave is partly re-reflected from the ground plane and partly transmitted into the transmission line. The magnitude and phase of this transmitted wave determines the impedance or admittance of the antenna. The re-reflected wave is again scattered and reflected from the end and the process is repeated, but let us first consider the effect of one reflected wave. This reflected wave is produced by an effective ring source at $z=L$

$$\bar{K}_e = -2V_e\delta(r-a)\delta(z-L)\hat{\phi}. \tag{28}$$

Let the electric and magnetic field from this source be denoted by $\bar{E}_e$ and $\bar{H}_e$ and let the fields from an incoming wave in the transmission line be denoted by $\bar{E}_A$ and $\bar{H}_A$. The value of $\bar{E}_e$ and $\bar{H}_e$ at the reference plane in the transmission line can be found by means of the reciprocity theorem. Following standard derivations (e.g., R. F. Harrington, 1961),

$$\int_S (\overline{H}_A \times \overline{E}_e - \overline{H}_e \times \overline{E}_A) \cdot \hat{n} dS = \int_V \overline{H}_A \cdot \overline{K}_e dV. \tag{29}$$

The surface S surrounding the volume V comprises the surface of the ground plane, the inside surface of the outer conductor in the transmission line, a transverse plane at $z = -n\lambda_0/2$, where the impedance is found, and a semi-infinite sphere. Due to the boundary conditions the only contribution comes from the transverse plane. The source of the fields $\overline{E}_A$ and $\overline{H}_A$ is assumed to be outside S. Introducing the voltage and current

$$\overline{E}_A = \frac{V_A}{r \ln \frac{b}{a}} \hat{r}, \tag{30}$$

$$\overline{H}_A = \frac{Y_\infty V_A}{2\pi r} \hat{\phi}, \tag{31}$$

The latter is a consequence of the definition of the admittance for the infinite antenna. Equations (30) and (31) give

$$I_A = Y_\infty V_A. \tag{32}$$

The $\overline{E}_e$, $\overline{H}_e$ fields constitute a wave traveling in the negative z directions, consequently

$$\overline{H}_e = \frac{C}{2\pi r} \hat{\phi}, \qquad \overline{E}_e = -\frac{C\zeta_0}{2\pi r} \hat{r}. \tag{33}$$

The unknown constant C is what we want to determine. Equation (33) gives a contribution to the current and voltage

$$I = C \qquad \text{and} \qquad V = -CZ_c, \tag{34}$$

where Z_c is the characteristic impedance of the line

$$Z_c = \frac{\zeta_0}{2\pi} \ln \frac{b}{a} \cdot \tag{35}$$

Insertion into (29) yields

$$C = \frac{2Y_w(L)e^{ik_0L}}{(1 + Z_cY_\infty)} V_e, \tag{36}$$

where the wave admittance is defined by (9).

We can now build up the solution from successive reflections. The incoming transmission line wave produces a gap voltage V_a and a current at the reference plane $I_A = V_A Y_\infty$; V_A produces an outgoing current wave

$$I_1^i(z) = Y_w(z)e^{ik_0z}V_A, \tag{37}$$

which excites an end-gap voltage V_{e1} found from (27)

$$V_{e1} = -\frac{Y_w(L)}{Y_e} e^{ik_0L} V_A, \tag{38}$$

which is the source for the first reflected wave

$$I_1^r = Y_w(L-z)e^{ik_0(L-z)}V_{e1}. \tag{39}$$

This wave is partially reflected and as a first approximation, the effective sources for the second outgoing wave must be the currents in the ground plane and the extra gap voltage $-C_1Z_c$ produced by the incident field. The currents in the ground plane are approximated by the image of V_{e1}. Thus we get the following for the second outgoing current wave

$$I_2^i = -Y_w(z)e^{ik_0z}Z_cC_1 + Y_w(L+z)e^{ik_0(L+z)}V_{e1}. \tag{40}$$

Equation (40) represents only an approximation, but as a first-order correction to the case of vanishing Z_c it should be correct. We can now repeat the steps taken for I_1^i and add up the result for the infinite number of reflections at the gap and at the end

$$V = V_A - Z_c(C_1 + C_2 + C_3 + \ldots)$$

$$I = V_AY_\infty + C_1 + C_2 + C_3 + , \tag{41}$$

where

$$C_n = \left[\frac{2Y_w^2(L)Z_ce^{i2k_0L}}{(1+Z_cY_\infty)Y_e} - \frac{Y_w(2L)e^{i2k_0L}}{Y_e}\right]C_{n-1}. \tag{42}$$

The reflections form a geometrical progression and the infinite series can be summed to give

$$Y' = \left(\sum_1^\infty C_n\right)\Big/V_A = \frac{-2Y_w^2(L)e^{i2k_0L}/(1+Z_cY_\infty)}{Y_e + Y_w(2L)e^{i2k_0L} - 2Y_w^2(L)\dfrac{Z_c}{(1+Z_cY_\infty)}e^{i2k_0L}} \tag{43}$$

where C_1 has been found from (36) and (38). The admittance of the antenna is then, from (41)

$$Y = \frac{I}{V} = \frac{Y_\infty + Y'}{1 - Z_cY'}. \tag{44}$$

Combining (43) and (44) gives

$$Y = Y_\infty - \frac{2Y_w^2(L)e^{i2k_0L}}{Y_e + Y_w(2L)e^{i2k_0L}}. \tag{45}$$

This is the steady-state solution that does not contain Z_c. The fact that Z_c cancels out serves as a verification of the validity of the calculations. An expression similar to (45) has been derived by Hallén (1953, 1962), who did not, however, consider the effect of the transmission line on the separate waves because he used the narrow-gap model for the symmetrical antenna.

Equation (45) is in principle valid for arbitrary lengths but if the evaluation of $Y_w(L)$ is limited to the asymptotic form (fig. 2, eq 8) k_0L should be somewhat larger than 1, depending on the thickness.

Formulas for very thin antennas can easily be derived from (45)

$$Y_\infty \sim \frac{2\pi}{\zeta_0}\frac{-1}{\ln\left(\dfrac{k_0a\Gamma}{2}\right)} - i0.$$

We have assumed a fixed gap size and very small $k_0 a$, so B_∞ can be neglected.

$$Y_w(L) \sim \frac{4\pi}{\zeta_0} \frac{1}{\ln \frac{iL}{l}}$$

$$Y_e \sim \frac{2\pi}{\zeta_0} \frac{-1}{\ln (k_0 a\Gamma/i)}$$

$$Y = \frac{2\pi}{\zeta_0} \frac{-1}{\ln \frac{k_0 a\Gamma}{2}} + \frac{4\pi}{\zeta_0 \ln \frac{a}{2L}} \cdot \frac{1}{\left(e^{-i2k_0 L} + 1 + \frac{\ln (4k_0 L\Gamma/i)}{2 \ln k_0 a}\right)}, \tag{46}$$

(a). $k_0 L \to 0$

$$G \sim \frac{2\pi}{\zeta_0} \frac{\ln (k_0 L\Gamma)}{\ln \frac{a}{2L} \cdot \ln \frac{k_0 a\Gamma}{2}}, \tag{47}$$

$$B \sim -\frac{2\pi}{\zeta_0} \frac{k_0 L}{\ln \frac{2L}{a}}. \tag{48}$$

Equation (47) indicates that the asymptotic forms should not be used for $k_0 L\Gamma < 1$ $\left(L < \frac{\lambda}{10}\right)$ in which case G is negative. Equation (48) is identical with $-2\omega C$, where C is the static capacitance of a thin ellipsoid of the same length and diameter (Weber, 1950).

(b). $k_0 L = \frac{\pi}{2}(1+2n), \qquad n = 0, 1, 2, \ldots.$

This is close to resonance points.

$$Y \sim \frac{-2\pi}{\zeta_0 \ln \left(\frac{k_0 a}{2}\right)} + \frac{8\pi}{\zeta_0} \frac{\ln ka}{\ln \frac{ka}{\pi(1+2n)}} \frac{1}{\ln (2\pi\Gamma(1+2n)) - i\frac{\pi}{2}}, \tag{49}$$

which for infinitely thin antennas gives

$$Z = 15 \ln (2\pi\Gamma(1+2n)) - i15\frac{\pi}{2} \text{ ohms},$$

$$n = 0, Z = 36.3 - i23.6 \text{ ohms}.$$

(c). $k_0 L = \pi(1+n), \qquad n = 0, 1, 2, \ldots.$
This is close to antiresonance points

$$Y \sim \frac{\ln k_0 L\Gamma}{60 \ln \left(\frac{k_0 a\Gamma}{2}\right) \ln \left(\frac{a}{2L}\right)} \text{ mhos}. \tag{50}$$

The examples given above serve only as an illustration of the theory; in any practical case (45) should be used directly with the various admittances taken from the curves.

In the admittance formula (45), only Y_W and G_∞ are known with any precision. Values of B_∞ and Y_e are only known in special cases and the accuracy has not been checked experimentally. But (45) gives a possibility for finding these quantities because the gap and the end effects are separated. The two unknowns Y_∞ and Y_e can be found by measuring Y at two arbitrary lengths; this gives two equations with two unknowns that are easily solved. The results of such a calculation are shown in table 1 and on figures 4 and 5.

The experimental basis is taken from the very precise measurements by Hartig (King 1956, pp. 239–250) and the arbitrary lengths are chosen as $k_0L=\frac{\pi}{2}$ and π. For Y_w the five-term series (10) given by Kunz is used. When $\frac{b}{a}$ is varied for a fixed k_0a, B_∞ varies rapidly and the other quantities remain approximately constant. A trend towards smaller values for G_∞ when $\frac{b}{a}$ increases can be noted in agreement with the results of section 2.2.1. In figures 4 and 5 mean values of G_∞, G_e, and B_e are used. We shall now discuss the various components.

1. Admittance of infinite antenna, Y_∞.

The conductance is in very good agreement with theoretical values so this serves as a verification of the results in section 2.2.1.

The susceptance varies in accordance with (24), but (24) can only be used qualitatively. On the same figure theoretical results of Hallén (1953) and Duncan (1962) are shown; these were discussed in section 2.2.2 and should be valid for a narrow gap, that is $\frac{b}{a}$ unspecified but close to one. If the transmission line is a standard antenna cable ($Z_c=70\ \Omega$), $\frac{b}{a}=3.2$, Duncan's values are too large and Hallén's values rather accurate in the range $0.01<k_0a<0.1$.

If Hartig's measurements are compared with King-Middleton's second-order theory it is seen that the latter implies a ratio of $\frac{b}{a}$ about 2 (see King, 1956 fig. 38.14). King (1956) stated that the difference between theory and experiment for thick antennas can be attributed to the fact that the theory is essentially a one-dimensional theory. From the results presented here the second-order theory appears to be accurate also for thicknesses as large as $k_0a=0.1$ for antennas with $\frac{b}{a}\sim 2$.

2. End admittance, Y_e.

In this case the comparison of theory and experiment is more difficult because no theory exists for the ends with hemispherical caps (Hartig's experiment) and no experiments exist for tubes with thin walls (Hallén's theory). Figure 5 shows that the conductance is larger for a spherical end and the difference is increasing with ka. This indicates that an increasing portion of power is radiated from the end which seems reasonable. The susceptance of the rounded end is smaller than the susceptance of the sharp-edged end which is a usual trend.

To check the accuracy for arbitrary lengths, we plotted (fig. 6) the resistance of the thickest antenna ($k_0a=0.1$) with $\frac{b}{a}$ as a parameter using the mean values of G_∞ and Y_e. The points denote Hartig's experimentally determined values. The accuracy is seen to be quite satisfactory.

4. Conclusion

The physics involved in radiation from a cylindrical wire is best understood in terms of the guiding and radiation of waves by the infinite cylinder. A thin cylinder guides the energy very effectively over a long distance until eventually all the energy is radiated; a thick cylinder is a more efficient radiator, the radiation damping with distance being greater. The phase velocity is

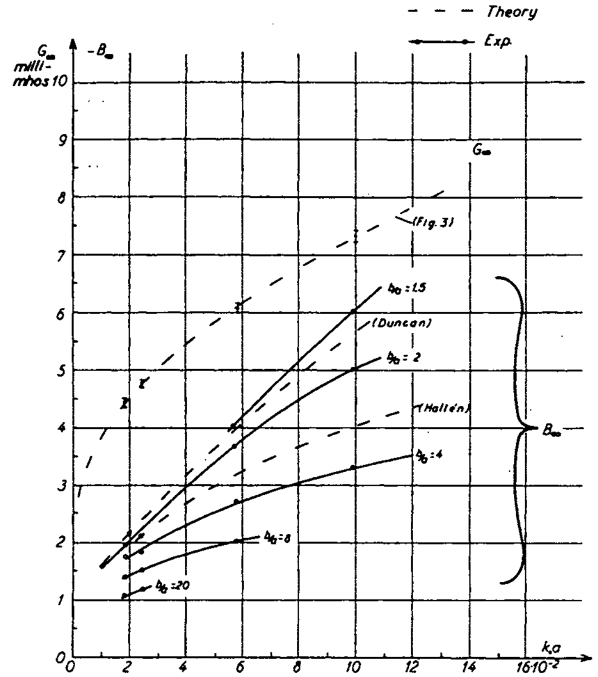

FIGURE 4. *Admittance of infinite antenna.*

The points are experimentally derived values. For the conductance the theoretical curve is taken from fig. 3. For the susceptance the theoretical curves are those of Hallén (1953) and Duncan (1962).

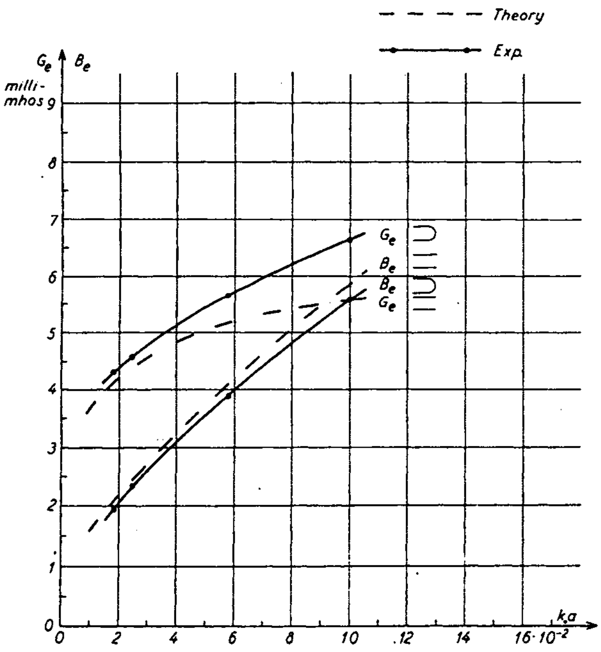

FIGURE 5. *End admittance.*

The experimentally derived values for an end with hemispherical cap are compared with theoretical values of Hallén for a thin-walled tube.

TABLE 1. *Experimental results of Hartig and the values of* Y_∞ *and* Y_e *derived from them.*

		$k_0L=\frac{\pi}{2}$		$k_0L=\pi$					
k_0a	b/a	R_1	X_1	R_2	X_2	G_e	B_e	G_∞	B_∞
		ohms				millimhos			
$1.872\cdot10^{-2}$	2.21	45.62	−21.63	253.8	340.4	4.340	−1.964	4.476	−1.915
	5.32	43.13	−21.37	311.3	347.1	4.303	−1.898	4.522	−1.644
	7.09	42.22	−21.28	360.7	350.7	4.296	−1.873	4.521	−1.441
	10.64	45.12	−23.12	401.3	359.5	4.352	−1.996	4.443	−1.255
	25.11	44.29	−24.95	464.7	377.3	4.301	−2.025	4.373	−1.053
$2.494\cdot10^{-2}$	1.67	48.21	−21.75	191.9	291.2	4.647	−2.363	4.790	−2.377
	4.00	44.78	−23.71	267.6	313.3	4.545	−2.322	4.830	−1.829
	5.33	45.08	−22.85	292.8	313.6	4.597	−2.309	4.827	−1.699
	8.00	44.36	−24.26	325.1	322.9	4.549	−2.330	4.799	−1.519
	18.88	45.54	−26.12	395.9	335.0	4.567	−2.426	4.704	−1.195
$5.818\cdot10^{-2}$	1.71	48.56	−18.01	110.1	187.2	5.632	−3.755	6.148	−3.864
	2.28	49.39	−19.13	127.0	197.1	5.662	−3.859	6.098	−3.452
	3.43	49.03	−21.12	160.0	205.5	5.629	−3.948	6.142	−2.864
	8.10	48.03	−21.86	232.1	223.6	5.729	−3.938	5.989	−2.006
$9.99\cdot10^{-2}$	1.33	52.67	−10.00	56.76	119.9	6.535	−5.570	7.450	−6.535
	2.00	52.19	−11.90	83.13	139.0	6.784	−5.661	7.311	−5.041
	4.72	49.02	−18.49	149.2	165.4	6.576	−5.665	7.203	−3.037

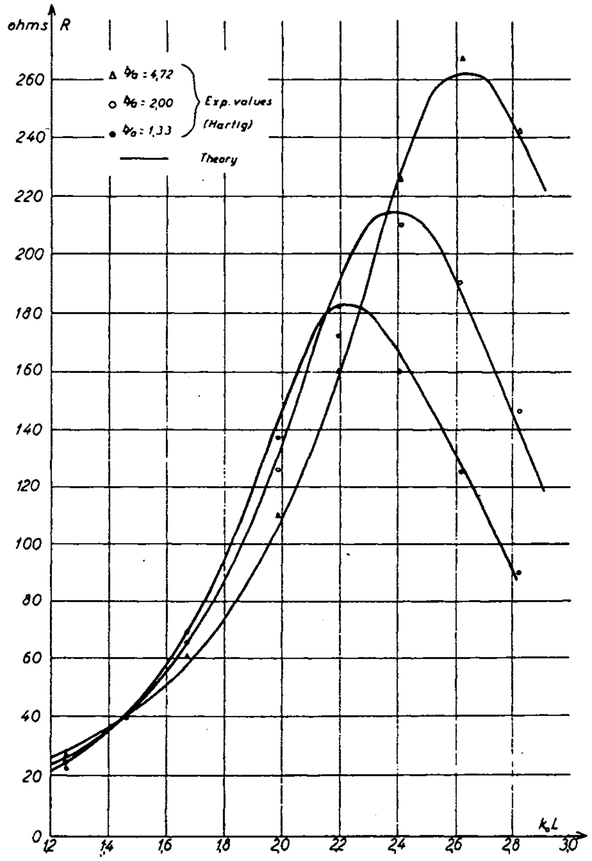

FIGURE 6. *Resistance of a thick antenna* ($k_0a=0.1$) *vs. length with gap size as a parameter.*

The points are experimental values (Hartig) and the curves are theoretical curves, eq (45).

slow compared with the velocity of light but varies asymptotically with distance approaching the velocity of light. Thus the wire is a nonuniform transmission line and this explains why it is radiating even though the wave is slow. The conductance of the infinite monopole over a ground plane has been found.

The standing-wave antenna is treated with an assumed effective source at the end that excites the reflected waves. The steady-state solution can then be found by summing up the infinity of multiply reflected waves. The coupling between the transmission line and the antenna is treated in detail and it is shown how exact values of the gap admittance for the infinite antenna can be derived from experimental data. Similarly the end admittance has been derived from experiment and compared with theoretical values given by Hallén for a thin-walled tube. In the range of parameters covered, very accurate values for the admittance of any monopole can be found.

The theory applies in principle equally well to dipole antennas fed by a two-wire transmission line. However, the symmetry of the excitation is destroyed and higher order waves will be excited; the radiation of antisymmetric waves will be treated theoretically and experimentally in a subsequent paper. The gap susceptance will be different but approximate values can be obtained by taking one-half the monopole susceptance for a gap of similar width.

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